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THE ELECTRICAL CONDUCTIVITY OF WEAKLY IONIZED GASES

Prepared by

Paul Molmud

SPACE TECHNOLOGY LABORATORIES, INC.
Redondo Beach, California

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Prepared:

Paul Molmud
Theoretical Physics Department

Approved:

Saul Altshuler
Manager
Theoretical Physics Department

Approved:

R. B. Muchmore
Director
Physical Research Division

QUANTUM PHYSICS LABORATORY
PHYSICAL RESEARCH DIVISION

Space Technology Laboratories, Inc.
One Space Park
Redondo Beach, California

ABSTRACT

Some elementary solutions of the wave equation are obtained for electromagnetic waves in an ionized gas. The solutions demonstrate the importance of conductivity in controlling absorption. Improper computation of conductivity by conventional theory may lead to order of magnitude errors in estimating absorption. The theory of A.C. conductivity for a weakly ionized gas is reviewed. It is shown when and how the exact theory may be employed in conventional form and without error. The expression for the conductivity of a weakly ionized mixture of gases is formulated and analyzed in the low, intermediate and high pressure regions. The conductivity of the mixture is shown to be reducible to the sum of conductivities of gases of constant cross section but complex wave frequencies. Appropriate tables for the evaluation of these latter conductivities are being constructed.

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THE ELECTRICAL CONDUCTIVITY OF WEAKLY IONIZED GASES

I. INTRODUCTION

Knowledge of the magnitude of absorption of radio waves by rocket exhausts and plasma sheaths is of vital concern in missile guidance and communication.

The conventional procedure for estimating the attenuation magnitudes is to compute or "determine" the electron density and "collision frequency" of the electrons in the gas in question. These parameters are then substituted into a conventional and over simplified relationship to yield the conductivity of the medium.

The use of improper methods of determining the electron collision frequency and computing conductivity may, however, lead to order of magnitude errors in the prediction of signal strength.

This paper will review the conventional, overly simplified theory of conductivity and then discuss the correct theory. We will see how, in certain cases, the simple theory may be recovered from the correct theory. In other cases the exact theory or certain modifications of it must be used.

The content of this paper is as follows: The wave equation of a conducting medium is solved for two elementary cases. The solutions demonstrate how conductivity enters into expressions of absorption of an electromagnetic wave. The theory of conductivity is then discussed in detail. Exact expressions for the conductivity of single component ionized gases and ionized mixtures of different gases are developed. On the basis of these results, the method of analysis of certain diagnostic measurements on rocket exhausts is criticized.

II. THE WAVE EQUATION AND SOME ELEMENTARY SOLUTIONS

A. Maxwell's Equations

In an ionized gas where we can neglect the polarizability of the gas molecules, Maxwell's equations for A.C. fields take the form

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{i\omega}{c} \vec{H} \\ \nabla \times \vec{H} &= \frac{i\omega}{c} \vec{E} + 4\pi \frac{\sigma}{c} \vec{E} \\ \nabla \cdot \vec{H} &= 0 \\ \nabla \cdot \vec{E} &= 4\pi\rho \end{aligned} \qquad 1)$$

Here ω is 2π times the wave frequency and σ is the conductivity of the medium.

B. The Wave Equation

By the usual manipulations of equations 1), we form the wave equation

$$\nabla^2 \vec{E} + \nabla \left(\vec{E} \cdot \frac{\nabla\sigma}{\sigma}\right) + \frac{\omega^2}{c^2} \left(1 - \frac{4\pi i}{\omega} \sigma \right) \vec{E} = 0 \quad . \qquad 2)$$

The only cases treated in this paper will be (.) σ constant and ·) $E \cdot \nabla\sigma = 0$, so that the second term vanishes.

C. Elementary Solutions of the Wave Equation

. σ Constant

We seek a solution of equation 2) for the case of constant conductivity and of the form $E = E_o e^{-ikz}$ (a plane wave propagating in the z-direction). When the propagation constant, k, is set equal to $\alpha - i\beta$, then the values of α and β consistent with 2) is given by:

$$\left\{ \begin{matrix} \alpha\lambda \\ \beta\lambda \end{matrix} \right\} = \pi\sqrt{2} \left\{ \left(1 + \frac{4\pi\sigma_i}{\omega}\right) \left[\left(1 + \left(\frac{4\pi\sigma_r}{\omega}\right)^2 \frac{1}{\left(1 + \frac{4\pi\sigma_i}{\omega}\right)^2}\right)^{\frac{1}{2}} \pm 1 \right] \right\}^{\frac{1}{2}} . \qquad 3)$$

Here λ is the wavelength (in vacuum) of the wave, σ_r and σ_i are the real and imaginary parts of the conductivity and the choice of the + or - sign determines whether the equation refers to α or β respectively. The absorption or attenuating factor is given by β and the phase factor by α. Thus: $E = E_o e^{-i\alpha z} e^{-\beta z}$.

When the following inequality is satisfied:

$$\left| \frac{4\pi\sigma_r}{\omega} \quad \frac{1}{\left(1 + \frac{4\pi\sigma_i}{\omega}\right)} \right| < 1, \qquad 4)$$

Then, we obtain the important approximations:

$$\alpha\lambda \approx 2\pi\left(1 + \frac{4\pi\sigma_i}{\omega}\right)^{\frac{1}{2}} \qquad 5)$$

$$\beta\lambda \approx \pi\left(\frac{4\pi\sigma_r}{\omega}\right) \frac{1}{\sqrt{1 + \frac{4\pi\sigma_i}{\omega}}} . \qquad 6)$$

The attenuation factor β, from 6), now varies as σ_r.

?. <u>σ Spatially Dependent, but $E \cdot \nabla\sigma = 0$</u>

We now seek a solution of 2) of the form $E = A(z)\, e^{i\phi(z)}$ where $\sigma = \sigma(z)$. (Compare with the Eikonal solution in, for example reference (2)).

The differential equations satisfied by A and ϕ are

$$(\frac{\partial \phi}{\partial z})^2 \quad \frac{1}{A} \frac{\partial^2 A}{\partial z^2} = \frac{\omega^2}{c^2} + \frac{4\pi\omega}{c^2} \sigma_i \quad , \tag{7}$$

$$(\frac{\partial^2 \phi}{\partial z^2}) + \frac{2}{A} \frac{\partial A}{\partial z} \frac{\partial \phi}{\partial z} = \frac{-4\pi\omega}{c^2} \sigma_r \quad . \tag{8}$$

We impose the requirement that $\frac{1}{A} \frac{\partial^2 A}{\partial z^2} << \frac{\omega^2}{c^2}$ (equivalent to demanding the amplitude of the wave not change significantly in a distance of one wavelength). Then:

$$\phi = \phi_o - \frac{\omega}{c} \int_{z_o}^{z} \sqrt{1 + \frac{4\pi\sigma_i(z')}{\omega}} \, dz' \quad . \tag{9}$$

Then combining 8) and 9), we obtain

$$\frac{A}{A_o} = \left[\frac{1 + \frac{4\pi}{\omega} \sigma_i(z_o)}{1 + \frac{4\pi}{\omega} \sigma_i(z)} \right]^{\frac{1}{4}} e^{-\int_{z_o}^{z} \frac{2\pi}{c} \frac{\sigma_r(z') \, dz'}{\sqrt{1 + \frac{4\pi\sigma_i}{\omega}}}} \quad . \tag{10}$$

When $\frac{4\pi}{\omega} \sigma_i << 1$, then E can be written as:

$$E = E_o \, e^{-i \int_{z_o}^{z} k(z')dz'} \quad , \tag{11}$$

where $k = \alpha(z) - i\beta(z)$ and α and β are given by 5) and 6). In this form, again, the absorption at each point varies as σ_r.

It is then seen that if errors are made in the computation of σ_r then extremely large errors can be made in E through equation 11). For example: suppose $\int_{z_o}^{z} \beta \, dz = 3$ and an error of 33 1/3% has been made in the computation of σ_r. Then $\int_{z_o}^{z} \beta \, dz$ can range from 4 to 2. Thus,

E^2 can range in magnitude from $E^2 = E_o^2\, e^{-4}$ to $E^2 = E_o^2\, e^{-8}$; a variation of about 50 in power.

III. CONDUCTIVITY THEORY

A. General Remarks

The electrical conductivity, σ, of any medium is defined by the equation

$$\vec{j} = \sigma \vec{E}\,, \qquad 12)$$

where $\vec{j}$ is the current density, σ is the conductivity (a scalar or tensor quantity) and $\vec{E}$ is the electric field vector. We may also describe $\vec{j}$ as due to transport of charge, thus

$$\vec{j} = \sum_i e_i\, \vec{v}_i / \Delta\tau \qquad 13)$$

where e_i and $\vec{v}_i$ are the charge and velocity, respectively, of the i^{th} charged particle and the summation is taken over all particles found in the infinitesimal volume element $\Delta\tau$ at which $\vec{j}$ is to be determined. In general, there are only a few species of particles (e.g., electrons, molecular ions) in the medium acting as charge carriers. We may thus rewrite equation 12) in the following form:

$$\vec{j} = \sum_k n_k\, e_k\, \bar{\vec{v}}_k \qquad 14)$$

where n_k is the particle density of the k^{th} species, e_k is the charge carried and $\bar{\vec{v}}_k$ is the <u>average</u> velocity of the k^{th} species. If $\bar{\vec{v}}_k$ is a linear function of $\vec{E}$ then σ can be described in terms of e_k, n_k and other quantities through equations 11) and 13).

We now consider a particular type of medium, that is a weakly ionized gas. By "weakly ionized" we mean that the only important disturbances the charged particles encounter other than that due to an externally imposed field, are those due to collisions with neutral gas particles.*

The charge carriers in this medium are positively and negatively charged molecules and free electrons. As shall be seen presently, $\tilde{v}_k \, \alpha (1/m_k)$. The current density in a gas containing free electrons will therefore be described mainly by the electron charge transport in the gas since molecular ions of the same concentration will contribute $\sim 1/40{,}000$ of this current density (assuming molecular weight of 20).

Mobility experiments in Bunsen burner flames have demonstrated that most of the current density there is indeed due to free electrons.(3) We shall assume that the same may be said about rocket flames. The denial of this assumption would result in an impossible degree of ionization required to explain the electromagnetic behavior of real rocket flames.

We will write $\vec{j}$, therefore, as

$$\vec{j} = n \, e \, \tilde{\vec{v}} \qquad 15)$$

where n is the electron density, e is the electronic charge and $\tilde{\vec{v}}$ the average electron velocity.

B. Conventional Conductivity Theory

Let us proceed, as is usually done, to assume a simple dependence of $\tilde{\vec{v}}$ on $\vec{E}$ and ν, the collision frequency between electrons and neutral particles. We do this to demonstrate how a functional form may be obtained for σ and also because we need the resulting form for subsequent considerations.

* As a rough rule, a gas is weakly ionized when the mole fraction of electrons is less than 10^{-4}.

We assume that $\tilde{v}$ satisfies the following dynamic equation of motion

$$m\dot{\tilde{\vec{v}}} + m\nu\tilde{\vec{v}} = e\vec{E}_o e^{i\omega t} \tag{16}$$

which is the same equation that describes the motion of a body undergoing forced oscillation in a viscous medium.

The steady state solution of equation 15) is

$$\tilde{\vec{v}} = \frac{e}{m} \frac{\vec{E}_o e^{i\omega t}}{i\omega + \nu} \tag{17}$$

the real part of which is

$$\tilde{\vec{v}}_r = \frac{e}{m} \vec{E}_o \left(\frac{\nu}{\omega^2 + \nu^2} \cos \omega t + \frac{\omega}{\omega^2 + \nu^2} \sin \omega t \right) . \tag{18}$$

The conductivity is obtained from equations 17), 15), and 12).

$$\sigma = \frac{ne^2}{m} \frac{1}{(i\omega + \nu)} . \tag{19}$$

This conductivity formula is only as valid as equation 15) is valid. The validity of equation 15) depends upon a very particular assumption made as to the nature of the interaction (collisions) between electrons and neutral particles. This assumption will be discussed later. To find the consequences of a general type of interaction requires a more exact conductivity theory.

C. The Correct Theory

The theory of conductivity for the case of free electrons undergoing elastic collisions with neutral particles has been established by Margenau.(4)

We write $\tilde{v}$, in this theory, in the form suggested by 17)

$$\tilde{\vec{v}} = \frac{e\vec{E}_o}{m} (B \cos \omega t + D \sin \omega t) \qquad 20)$$

and B and D are defined below and $\sigma = \frac{ne^2}{m} (B - iD)$.

B and D (for the weak field* case only) has the form:

$$B - iD = \frac{8\pi}{3} \beta\left(\frac{\beta}{\pi}\right)^{3/2} \int_o^{\infty} \frac{v^4 e^{-\beta v^2} dv}{i\omega + \nu} \qquad 21)$$

where $\beta = m/(2kT)$, k being the Boltzmann constant and T the temperature of the gas. Also, ν is now the collision frequency for electrons of a given velocity, v, with the neutral molecules. This collision frequency will be represented as

$$\nu(v) = \sum_i \rho_i Q_i(v)v, \qquad 22)$$

where ρ_i is the numerical particle density of the i^{th} molecular species, $Q_i(v)$ is the elastic cross section for momentum transfer, a measure of collision probability between the electron and an i^{th} species molecule and is, in general, a function of electron velocity, v.

It has been shown by Altshuler (5), (6) that inelastic cross sections play the same role in conductivity theory as elastic ones provided the energy changes on collision are small relative to the energy of the electron. Thus, the Q_i's could represent inelastic as well as elastic cross sections when needed. However, in energy regions we are interested in, $< \frac{1}{2}$ eV, the inelastic cross sections are considerably smaller than elastic.(7)

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* i.e., the energy gained by an electron from the field during a mean free time is less than the energy lost upon collision.

We shall now develop expressions for conductivity obtained by using realistic cross sections $Q_1(v)$.

D. Conductivity of Some Gases Having Cross Sections with Simple Dependence on Electron Velocity

We now determine the functional forms of the conductivity when electrons are colliding with a single species of gas molecule and the cross section has the particularly simple dependence on velocity, $Q = cv^j$. Here c is a constant and j is an integer.

1.) $j = -1$

The first gas we treat as one where $Qv = c$ (a Maxwellian gas(8)). Then equation 21) can be integrated very readily and we obtain

$$B - iD = \frac{1}{i\omega + \tilde{\nu}}, \qquad 23)$$

yielding the same expression for conductivity as in 19). Here $\tilde{\nu}$, the average collision frequency is given by ρQv.

2.) j an Integer

The electron momentum-transfer cross sections for many gases have been experimentally determined. The cross section for several in the temperature range between .04 ev and 1 ev may be represented by $Q = cv^j$ where j is an integer (9), and c a constant whose value depends on the molecular species. For example, $j = 1$, describes Nitrogen; $j = 0$, Helium, $j = -2$, H_2O, NH_3, HCl (molecules with permanent dipole moments (10)). Also, $j = -1$ appears to represent the cross section of CO_2 (11). Table I presents some values of cross sections for some typical gases found in rocket exhausts.

TABLE I

Momentum transfer cross sections of electrons impacting on several molecular species.

(v is velocity of electrons)

Species	Cross Section (c.g.s.)	Reference	Temperature range
CO	$Q = 2.08 \times 10^{-23} v + 2.64 \times 10^{-16}$	(11)	$T \lesssim 2.5 \times 10^4 {}^\circ K$
CO_2	$Qv = 4.7 \times 10^{-8}$	(11)	$T \lesssim 10^4 {}^\circ K$
H_2O	$Qv^2 = 5.9$	(10)	$T \lesssim 10^4 {}^\circ K$
HCl	$Qv^2 = 1.85$	(10),(13)	$T \lesssim 10^4 {}^\circ K$
NH_3	$Qv^2 = 3.7$	(10)	$T \lesssim 10^4 {}^\circ K$
N_2	$Q = 3.29 \times 10^{-23} v$	(14)	$288^\circ K < T < 1200^\circ K$
H_2	$Q = 1.45 \times 10^{-23} v + 8.9 \times 10^{-16}$	(15)*	$300^\circ K \lesssim T \lesssim 10^4 {}^\circ K$
H_2	$Q = 8 \times 10^{-16}$	(16)	$50^\circ K \lesssim T \lesssim 5000^\circ K$

* Straight line approximation

The integrals arising from 21) upon employing the relationship $Q = cv^j$ can be reduced to tabulated functions for all of the values of j mentioned above. We demonstrate this for $j = 0$.

For $j = 0$:

$$B - iD = \frac{8\pi}{3}\ \beta\left(\frac{\beta}{\pi}\right)^{3/2} \int_0^{\infty} \frac{v^4 e^{-\beta v^2}\, dv}{i\omega + \rho Q v}\ , \tag{24}$$

where now Q is a constant. Upon using the transformation $\beta v^2 = \epsilon$, we obtain

$$B - iD = \frac{4}{3\sqrt{\pi}}\ \frac{\beta}{\rho Q} \int_0^{\infty} \frac{\epsilon^{7/2} e^{-\epsilon}\, d\epsilon}{\frac{i\omega\sqrt{\beta}}{\rho Q} + \epsilon^{\frac{1}{2}}} = \frac{4}{3\sqrt{\pi}}\ \frac{\beta}{\rho Q}\left[\int_0^{\infty} \frac{\epsilon^{4} e^{-\epsilon}\, d\epsilon}{\frac{\omega^2\beta}{\rho^2 Q^2} + \epsilon} - \frac{i\omega\sqrt{\beta}}{\rho Q} \int_0^{\infty} \frac{\epsilon^{7/2} e^{-\epsilon}\, d\epsilon}{\frac{\omega^2\beta}{\rho^2 Q^2} + \epsilon} \right] . \tag{25}$$

Both of these integrals in 25) may be found tabulated in reference (12). (c.f. the functions $\mathcal{U}_p(x)$ in this reference, with $p = 4$ and $p = 3.5$.) These integrals may be further reduced (9) to the form:

$$B - iD = \frac{4x^{\frac{1}{2}}}{3\sqrt{\pi}\,\omega}\left[1 - x - x^2 e^x E_1(-x)\right] - i\,\frac{4}{3\sqrt{\pi}\,\omega}\ \frac{x}{\omega}\left[\left(\tfrac{1}{2} - x\right)\sqrt{\pi} + \pi\, x^{3/2} e^x\left(1 - \phi(\sqrt{x})\right)\right] , \tag{26}$$

where $x = \frac{4\omega^2}{\pi \bar{\gamma}^2}$,

$\bar{\nu} = \frac{2\rho Q}{\sqrt{\pi\beta}}$, the average collision frequency,

$-E_1(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$ (the exponential integral),

$\phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt$ (the error integral).

The average collision frequency, $\tilde{\nu}$, is rigorously defined as follows: $\tilde{\nu} = \int \rho Q v f d\tau$, where f is the normalized velocity distribution function (assumed Maxwell-Boltzmann) and integration is overall of velocity space.

Transformation of the Correct Conductivity Theory to the Conventional Forms

The expressions 23) and 26), are obviously not equivalent and the use of 23) instead of 21) will, in general, produce wrong results.

There is a way, however, of correctly computing conductivity using the useful form of 23) but with some minor modifications.

We proceed as follows: Since 23), as it stands, is incorrect for $j \neq -1$, we try, for these j's, to force the correct expression of conductivity into the following form.

$$\sigma = \frac{ne^2}{m} \frac{1}{i\omega + g} , \tag{27}$$

where g is to be determined by equating 27) to

$$\sigma = \frac{ne^2}{m} (B - iD) , \tag{28}$$

and B and D are obtained from 21).

Now 27) and 28) are compatible only when either

1) $B = \frac{\omega}{\omega^2 + g^2}$ and $D = \frac{g}{\omega^2 + g^2}$

or

ii) g is complex, that is $g = g_r + ig_i$ and

$$g_r = B/(B^2 + D^2)$$
$$g_i = D/(B^2 + D^2) - \omega \quad . \tag{29}$$

i) is true only for a Maxwellian gas (i.e., $j = -1$). For this case, $g = \tilde{\nu}$, the average collision frequency, and $g_i = 0$.

ii) holds for all other cases.

Thus, in general

$$\sigma_r = \frac{ne^2}{m} \frac{g_r}{(\omega + g_i)^2 + g_r^2} \text{ and } \sigma_i = -\frac{ne^2}{m} \frac{(g_i + \omega)}{(\omega + g_i)^2 + g_r^2} \quad . \tag{30}$$

When a static magnetic field is present σ becomes a tensor. The components of this tensor are very similar to 30). A full discussion of this case may be found in reference (9).

As we have seen, B and D can have some very complicated forms (cf. 26). Therefore, no apparent simplification arises from a description of conductivity in the manner of 30) unless possibly for certain conditions where g is essentially real. Now equation 30) can be considered to arise from a differential equation as in 16) but with ν replaced by g. This latter equation and its resultant solution have intuitively satisfying forms. The desire to retain these forms motivates us to look for conditions under which g_i may be neglected. Further motivation is provided by the fact of the existence of a tremendous amount of analysis based on 16), already in the literature (e.g., the Appleton-Hartree formula). Under conditions where g_i may be neglected, the propagation formulae in literature may be corrected by the formal substitution of g_r for ν, whenever ν appears. More generally, in the literature, the expressions for conductivity may be rectified by formally replacing ω by $\omega + g_i$ and ν by g_r.

An inspection of the asymptotic forms of B and D for the various J's discussed above, reveals that g_i may be neglected under the following limiting conditions:

$$\frac{\tilde{\nu}}{\omega} \ll 1, \text{ the low-pressure case;}$$

$$\frac{\tilde{\nu}}{\omega} \gg 1, \text{ the high-pressure case.} \qquad 31)$$

The limiting forms of g_r and g_i for these two cases* have been evaluated (9) and are presented in Table II.

It is to be observed that g_r in the low-pressure limit is a simple multiple of $\tilde{\nu}/3$, and g_i is of the order $\tilde{\nu}^2/\omega^2$ with respect to ω. Therefore, it is reasonable to ignore g_i in equation 31) for $\frac{\tilde{\nu}}{\omega} < 1$. On the other hand, g_i in the high-pressure limit, especially for air, may be of the same order as ω and cannot be dropped unless ω itself can be neglected.

Note that according to the last column of the table, g_r is temperature dependent.

We can already see the magnitudes of the errors one can make in the incorrect formulation of conductivity. For example, in the low-pressure

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* Values of g_r and g_i vs $\frac{\tilde{\nu}}{\omega}$ for air at all pressures have recently been published and are available in reference (19).

TABLE II

Evaluation of g_r and g_i for the low- and high-pressure cases

Cross Section Dependence and Molecular Species	$\frac{\tilde{\nu}}{\omega} \ll 1$		$\frac{\tilde{\nu}}{\omega} \gg 1$		Form of Average Collision Frequency *
	g_r	g_i	g_r	g_i	$\tilde{\nu}$
$\frac{Q}{v}$ constant, air, nitrogen	$\frac{5}{3}\tilde{\nu}$	$\frac{10}{9}\ \frac{\tilde{\nu}^2}{\omega}$	$\tilde{\nu}$	2ω	$\tilde{\nu} = \frac{3Q}{v}\ \frac{\rho kT}{m}$
Q constant, helium (possibly hydrogen)	$\frac{4}{3}\tilde{\nu}$	$0.18\ \tilde{\nu}^2/\omega$	$\frac{3\pi\tilde{\nu}}{8}$	0.18ω	$\tilde{\nu} = 2\rho Q(2kT/\pi m)^{\frac{1}{2}}$
Qv constant, Maxwellian gas (possibly CO_2)	$\tilde{\nu}$	0	$\tilde{\nu}$	0	$\tilde{\nu} = \rho Q v$
Qv^2 constant, H_2O, NH_3, HCl, HF; molecules with permanent electric dipole moments.	$\frac{2}{3}\tilde{\nu}$	$0.079\ \tilde{\nu}^2/\omega$	$\frac{3\pi\tilde{\nu}}{16}$	0.11ω	$\tilde{\nu} = 2Qv^2\rho(\frac{m}{2\pi kT})^{\frac{1}{2}}$

* $\tilde{\nu}$ is rigorously defined as follows: $\tilde{\nu} = \int \rho Q(v) v f d\tau$, where ρ is the neutral particle density, $Q(v)$ is the cross section with its correct functional dependence on v, f is the normalized distribution (assumed Maxwell-Boltzmann) and integration is over velocity space.

limit for weakly ionized air, the use of equation 19) to predict absorption instead of 28), would result in errors (in db) of 67% !

F. Conductivity of Some Gases with Cross Sections Expressible as Polynomials in v and Also Conductivity of Mixtures of Gases

1 CO, Carbon Monoxide

Let us first consider the conductivity of weakly ionized CO The cross section for CO may be expressed as $Q = a + bv$ ((11) and Table I)., where a and b are constants. Then,

$$\sigma = \frac{8\pi}{3}\beta\left(\frac{\beta}{\pi}\right)^{3/2}\int_0^\infty \frac{v^4 e^{-\beta v^2}dv}{i\omega + \rho a v + \rho b v^2}, \tag{32}$$

which, by the method of partial fractions, may be expressed as:

$$\sigma = \frac{8\pi}{3}\beta\left(\frac{\beta}{\pi}\right)^{3/2}\frac{ne^2}{m\rho b}\frac{1}{\sqrt{\frac{a^2}{b^2} - \frac{4i\omega}{\rho b}}}\left[\int_0^\infty \frac{v^4 e^{-\beta v^2}dv}{\frac{a}{2b} - \sqrt{\frac{a^2}{4b^2} - \frac{i\omega}{\rho b}} + v} - \int_0^\infty \frac{v^4 e^{-\beta b^2}dv}{\frac{a}{2b} + \sqrt{\frac{a^2}{4b^2} - \frac{i\omega}{\rho b}} + v}\right] \tag{33}$$

Now let us make the following formal identification:

$$\frac{n}{\sqrt{\frac{a^2}{b^2} - \frac{4i\omega}{\rho b}}}\int_0^\infty \frac{v^4 e^{-\beta v^2}dv}{\frac{\rho a}{2} - \rho b\sqrt{\frac{a^2}{4b^2} - \frac{i\omega}{\rho b}} + \rho b v} = n'\int_0^\infty \frac{v^4 e^{-\beta v^2}dv}{i\Omega + \rho b v} \tag{34}$$

where $i\Omega = \frac{\rho a}{2} - \rho b\sqrt{\frac{a^2}{4b^2} - \frac{i\omega}{\rho b}}$ and $n' = \frac{n}{\sqrt{\frac{a^2}{b^2} - \frac{4i\omega}{\rho b}}}$. With reference to 24) expression 34) may now be identified as the A.C. conductivity of a gas of constant cross section, b, at a wave frequency Ω , but here Ω is complex and with an electron density n', also complex. Thus, the solution

26) is also valid for 34) but now ω is formally replaced by Ω, and ν by b.

The second integral of 33) is, as the first integral, identified as the A.C. conductivity of a gas of constant cross section for a wave frequency of Ω' where $i\,\Omega'$ is $\frac{a}{2b} + \sqrt{\frac{a^2}{4b^2} - \frac{i\omega}{\rho b}}$.

Thus, the conductivity of weakly ionized carbon monoxide may be represented as the difference of the conductivities of a gas of constant cross section evaluated at two different complex wave frequencies.

The evaluation of 32) is best done, at present, by numerical methods. Tables of values of integrals of the form of the right hand of 34), for complex Ω are being prepared at Space Technology Laboratories and will be available shortly. The integral of 34) may also be evaluated albeit tediously, with existing tables. In 26), since x is complex, error functions and exponential integrals of complex argument must be evaluated. Tables of such functions do exist. The tables of error functions of complex argument (Fried and Conte (17)) is readily available. However, the only tables of exponential integrals of complex argument known to me (Mashiko (18)) is difficult to obtain.

2. <u>A Mixture of Gases</u>

Let us suppose that the cross section for each gas in a mixture may be represented by a polynomial in v, $P_i(v)$ (where $P_i(v) = \sum_{j=0}^{n_i} a_j v^j$). Then the conductivity of a weakly ionized mixture is:

$$\sigma = \frac{8\pi}{3}\,\rho\left(\frac{\beta}{\pi}\right)^{3/2}\frac{ne^2}{m}\int_0^\infty \frac{v^4 e^{-\beta v^2}\,dv}{i\omega + v\sum_i P_i(v)\rho_i}\,. \qquad 35)$$

a The Low-Pressure Limit

In the low-pressure limit (i.e., $\omega >> v \sum_i P_i(v)\rho_i$ for important v) then 35) becomes.

$$\sigma \approx \frac{8\pi}{3} \beta\left(\frac{\beta}{\pi}\right)^{3/2} \frac{ne^2}{m} \int_0^\infty \frac{(v \sum_i P_i(v)\rho_i - i\omega)}{\omega^2} v^4 e^{-\beta v^2} dv \ . \qquad 36)$$

But $\frac{8\pi}{3} \beta\left(\frac{\beta}{\pi}\right)^{3/2} \frac{ne^2}{m} \int_0^\infty \frac{vP_i(v)\rho_i v^4 e^{-\beta v^2} dv}{\omega^2}$ is the real part of the conductivity of a low-pressure weakly ionized gas where the electrons collide with only the i^{th} species of molecule. Thus, we arrive at the following result: In the low-pressure limit, the real part of the conductivity of a weakly ionized mixture of gases is the sum of the individual real conductivities of each weakly ionized species of gas.

b. The High-Pressure Limit $\omega < \sum_i \rho_i P_i(v)v$

In the high-pressure limit,

$$\sigma \approx \frac{8\pi}{3} \beta\left(\frac{\beta}{\pi}\right)^{3/2} \frac{ne^2}{m} \int_0^\infty v^4 e^{-\beta v^2} \frac{\left[-i\omega + \sum_i P_i(v)\rho_i v\right]}{\left(\sum_i \rho_i P_i(v)v\right)^2} dv \ , \qquad 37)$$

and the conductivity is no longer simply related to the individual conductivities of the component gas. However, as for the case of carbon monoxide, we can show that σ, in the high pressure and intermediate pressure regions, is the sum of the conductivities of a gas of constant cross section evaluated at n different complex wave frequencies. This proof follows.

σ in General

When $\omega \lesssim \sum_i \rho_i P_i(v)v$, then we proceed as follows: $\sum \rho_i P_i(v)v = P(v)$ a polynomial of degree n. Then,

$$\sigma = \frac{8\pi}{3} \beta\left(\frac{\beta}{\pi}\right)^{3/2} \frac{n\epsilon^2}{m} \int_0^\infty \frac{v^4 e^{-\beta b^2} dv}{i\omega + P(v)} \quad . \tag{38}$$

Now $i\omega + P(v) = 0$ has n different* complex roots, $c_1, c_2, c_3 \ldots, c_n$. Thus, by the method of partial fractions:

$$\sigma = \frac{8\pi}{3} \beta\left(\frac{\beta}{\pi}\right) \frac{ne^2}{m} \int_0^\infty v^4 e^{-\beta b^2} dv \sum_{j=1}^{n} \frac{a_j}{v - c_j} \quad . \tag{39}$$

Thus, we arrive at the following general statement: <u>The conductivity of a weakly ionized mixture of gases may be interpreted as the sum of individual conductivities of gases of constant cross section but evaluated at various complex wave frequencies, ic_j, and complex electron densities na_j.</u>

Thus, if the individual cross sections pertaining to each molecular species are known, the conductivity may be evaluated by 38). If the proper tables are available σ may be obtained through equation 39).

IV. DIAGNOSTICS EXPERIMENTS

Occasionally for some ionized gases there is sufficient uncertainty in knowledge of cross sections and composition to necessitate the measurement of collision frequency as well as electron density. A common procedure, in the case of rocket exhausts, is to determine the absorption of microwaves of several frequencies due to the exhaust. The variation of absorption with frequency is then analyzed for a "collision frequency" and electron density. We now examine this procedure and discuss some pitfalls.

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*Since $i\omega + P(v)$ and its derivative with respect to v have no common factor then all the roots are different. See H. B. Fine, A College Algebra (Ginn and Co. 1904, p. 463, 199).

The absorption per unit length of exhaust is given by β of equation 6). Now suppose, as is often the case that $\frac{4\pi\sigma_i}{\omega} << 1$, and also, as is not often the case, that the exhaust gas is composed of Maxwellian molecules (i.e., Qv = constant). Then according to 23) and 6),

$$\beta = \frac{1}{2c} \frac{\omega_p^2 \tilde{\nu}}{(\omega^2 + \tilde{\nu}^2)} \qquad 40)$$

where $\omega_p^2 = 4\pi \frac{ne^2}{m}$, the square of the plasma frequency, $\tilde{\nu}$ the average collision frequencies and $\omega = 2\pi f$.

When β displays significant frequency dependence, then two measurements of attenuation per unit length (say β_1 and β_2) at two different frequencies (ω_1 and ω_2) allow for the determination of both $\tilde{\nu}$ and ω_p^2. For example:

$$\frac{\beta_1}{\beta_2} = \frac{\omega_1^2 + \tilde{\nu}^2}{\omega_2^2 + \tilde{\nu}^2} \qquad 41)$$

and $\tilde{\nu}$ maybe determined and then ω_p^2.

We now remind ourselves that the gases in rocket exhausts are not Maxwellian gases. What does this imply for a procedure such as 41)? For this case, we have shown (cf. equation 30)) that for certain pure gases that the nearest equivalent to 41) is:

$$\frac{\beta_1}{\beta_2} = \frac{(\omega_1 + g_{i1})^2 + g_{r1}^2}{(\omega_2 + g_{i_2})^2 + g_{r2}^2} \qquad 42)$$

and g_{i1} and g_{r1} are functions $\frac{\tilde{\nu}}{\omega_1}$ and g_{i2} and g_{r2} are also functions of $\frac{\tilde{\nu}}{\omega_2}$, where $\tilde{\nu}$ is the rigorously defined average collision frequency.

Equation 42) can be solved for $\tilde{\nu}$ since all the g's are functionally related. However, I feel in this case, it is much better to solve for $\tilde{\nu}$ directly from the general formulation of σ_r than to go the round about way of the g's. In the example of 42), if the gas were solely of constant cross section than from 26)

$$\frac{\beta_1}{\beta_2} = \frac{\left[1 - x_1 - x_1^2 e^{x_1} E_i(-x_1)\right]}{\left[1 - x_2 - x_2^2 e^{x_2} E_i(-x_2)\right]} \quad , \qquad 43)$$

where $x = \frac{4\omega^2}{\pi \tilde{\nu}^2}$. Thus, $\tilde{\nu}$ can be solved in terms of the experimentally determined ratio $\frac{\beta_1}{\beta_2}$.

If one persists in solving equations of the form 41), then the magnitude of the " ν " that is obtained will depend on the frequencies ω_1 and ω_2 and will not represent an invariant characteristic of the exhaust gases.

Another pitfall in the use of 41) is that the temperature dependence of ν is likely to be ignored. If the true $\tilde{\nu}$ is to be used to compute conductivities in an expanding gas, the temperature as well as the density of the gas must be considered. In nitrogen, for example, (Table II) the collision frequency varies as the pressure.

An additional possible error for N_2 is that g_r is $\tilde{\nu}$ at high-pressures and $\frac{5}{3}\tilde{\nu}$ at low-pressures. Thus, the effective collision frequency (for conductivity) of N_2 has a more complicated dependence than simply as the pressure.

Rocket exhaust gases are generally composed of several molecular species. Sometimes the cross section of one species is so much higher than the others that the electron collisions may be considered to be effectively only with that species. This is the case for lox-alcohol and lox RP1 where the dominant species in the exhaust is H_2O. With solid propellant exhausts there does not seem to be one dominant species. Under the latter circumstances it becomes very difficult to interpret multifrequency absorption measurements in the rigorous theory 35). The purpose of these measurements should then be to determine the weighted (by mole fraction) cross sections of each of the constituents of the exhaust as well as the electron density. If there are n parameters describing the weighted cross sections then n + 1 experiments would have to be done on the exhaust to enable inversion of 35) for the important parameters (including electron density).

I believe the following procedure is more satisfactory than 41) when making measurements on a mixture:

(1.) Assume the cross sections are known (as in Table I for example.)

(2.) Assume the composition is known from thermodynamic considerations.

(3.) Compute the A.C. conductivity from 38) or 39).

(4.) Compare absorption measurements at two different frequencies for consistency.

(5.) If inconsistent, adjust the weighted cross section of the species known least reliably until consistency is achieved.

(6.) Perform a third measurement at a third frequency and check for consistency in the predicted attenuations.

(7.) If there is still no consistency, adjust the weighted cross sections of the two species known least reliably until consistency among all three measurements is obtained.

V. SUMMARY

It has been demonstrated that errors in the determination of the conductivity of a weakly ionized gas may lead to errors of many orders of magnitude in absorption. One type of error, which can be avoided, is the incorrect formulation of the expression for conductivity. As demonstrated in this paper, inaccurate formulation may produce substantial errors, even as large as 67%, in conductivity. This inaccuracy may be catastrophic, particularly under conditions of heavy absorption.

The exact theory of conductivity may be employed very simply under conditions of low-pressure for all molecular species and at high-pressures for those molecules whose cross sections obey the simple law Qv^{-j} = constant.

The conductivity of a weakly ionized mixture of gases has also been analyzed. At low pressures the real part of the conductivity is equal to the sum of the partial conductivities, treating each species as a separate ionized gas. When the cross section for each species may be written as a polynomial in velocity (i.e., $Q_i = \sum_{j=0}^{n_i} a_j v^j$) then, for any pressure, the conductivity of the mixture has been shown to be the sum of the conductivities of gases of constant cross section, each one evaluated at a different complex wave frequency. Tables of the appropriate functions to evaluate such partial conductivities are now being prepared at Space Technology Laboratories.

Finally, caution is urged upon those who are performing diagnostic experiments on exhausts and plasma sheaths to analyze their data with care and consistency, using the correct expressions for conductivity.

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by Paul Molmud
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Unclassified Report

Some elementary solutions of the wave equation are obtained for electromagnetic waves in an ionized gas. The solutions demonstrate the importance of conductivity in controlling absorption. Improper computation of conductivity by conventional theory may lead to order of magnitude errors in estimating absorption. The theory of A.C. conductivity for a weakly ionized gas is reviewed. It is shown when and how the exact theory may be employed in conventional form and without error. The expression for the conductivity of a weakly ionized mixture of gases is formulated and analyzed in the low, intermediate and high pressure regions. The conductivity of the mixture is shown to be reducible to the sum of conductivities of gases of constant cross section but complex wave frequencies. Appropriate tables for the evaluation of these latter conductivities are being constructed.

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